A 63-dimensional counterexample to Borsuk's conjecture

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Abstract

We exhibit a set of 321 points in $\mathbb{R}^{63}$ that cannot be partitioned into 64 subsets of smaller diameter, so Borsuk's conjecture fails in dimension 63. The previous smallest counterexample was in dimension 64 (Jenrich, 2014), and the conjecture was open for $4 \leq n \leq 63$. The construction adds a single point to the 320-point rank-63 subconfiguration of Bondarenko's two-distance set, whose counting bound has been stuck at exactly $\lceil 320/5 \rceil = 64$ parts. The added point is not a vertex of the underlying strongly regular graph, so the resulting set is a three-distance set; this is precisely why the example was not reachable inside the two-distance framework in which all previous work took place.

1 Setting

Let G be the $G_2(4)$ graph, a strongly regular graph with parameters $(416, 100, 36, 20)$ and spectrum 100^1, 20^{65}, $(-4)^{350}$. It is well known (and re-verified here) that $\omega(G) = 5$.

Let E denote the orthogonal projection onto the 20-eigenspace of the adjacency matrix A, a space of dimension 65, and let $x_i = Ee_i$, $i \in V(G)$, be the 416 representing points. They form Bondarenko's two-distance set [1]:

$$\|x_i\|^2 = \tfrac{5}{32}, \qquad \langle x_i, x_j \rangle = \begin{cases} \frac{1}{32}, & i \sim j, \\ -\frac{1}{96}, & i \not\sim j, \end{cases}$$

so the two squared distances are $a^2 = \frac{1}{4}$ (adjacent) and $b^2 = \frac{1}{3}$ (non-adjacent). Thus b is the diameter, the diameter graph is the complement of G, and a subset has diameter smaller than b if and only if it is a clique of G. Bondarenko's counterexample in $\mathbb{R}^{65}$ is the statement $\lceil 416/5 \rceil = 84 > 66$; Jenrich's in $\mathbb{R}^{64}$ [2] deletes the support of a weight-64 vector of the eigenspace, leaving 352 points of rank 64 with $\lceil 352/5 \rceil = 71 > 65$.

Write $\mathcal{C} = \ker(A - 20I)$ for the eigenspace, regarded as a code of length 416; for $u \in \mathcal{C}$ we have $\langle x_i, u \rangle = (Eu)_i = u_i$, so for $Z \subseteq V(G)$

$$\operatorname{span}\{x_i : i \notin Z\} = \{u \in \mathcal{C} : \operatorname{supp}(u) \subseteq Z\}^{\perp} \cap \mathcal{C}. \tag{1}$$

2 The configuration that ties

Fix a two-dimensional subspace $P \subseteq \mathcal{C}$ spanned by two minimum-weight ($= 64$) words whose supports meet in 32 positions, so that its support $W = \bigcup_{u \in P} \operatorname{supp}(u)$ has $|W| = 96$. Such planes exist in abundance: we know 4809 of them, 65 "split planes" and 4744 found by information-set decoding. (Empirically 96 is also the second generalized Hamming weight of $\mathcal{C}$, but nothing below uses that; all that is needed is one plane of support 96.) Put

$$S = V(G) \setminus W, \qquad |S| = 320.$$

By (1), $\operatorname{span}\{x_i : i \in S\} = P^\perp$ has dimension 63. One checks $\omega(G[S]) = 5$, so the counting bound gives only

$$\left\lceil \tfrac{320}{5} \right\rceil = 64 = \dim + 1,$$

an exact tie: no contradiction. The tie is not an artifact of the bound. The graph $G[S]$ really does admit a partition into 64 disjoint 5-cliques (found by CP-SAT; the fractional relaxation of the clique cover is exactly 64.0), so its clique cover number is exactly 64 and no sharpening of the chromatic estimate can help. Deleting further vertices only decreases $|S|$. Inside the two-distance framework this configuration is a dead end, and it has been one since 2014.

3 The phantom point

Fix any $x_0 \in W$. Note that x_0 is a *deleted* vertex, so x_0 is not one of the 320 points of S. Let

$$v = \operatorname{proj}_{\operatorname{span} S}(x_0), \qquad y = \mu v,$$

where μ is the positive root of

$$\|v\|^2 \mu^2 - 2p_{\text{far}}\, \mu + \left(2p_{\text{far}} - R^2\right) = 0, \qquad R^2 = \tfrac{5}{32}, \quad p_{\text{far}} = -\tfrac{1}{96}. \tag{2}$$

Set $X = \{x_i : i \in S\} \cup \{y\}$, so $|X| = 321$.

Lemma 1. $\|v\|^2 = \frac{13}{96} > 0$, *for every $x_0 \in W$ and every such plane P.*

Proof. Since $\langle x_i, u\rangle = u_i$ for $u \in \mathcal{C}$, we have $\|\operatorname{proj}_P(x_0)\|^2 = w^\top \Gamma^{-1} w$, where Γ is the Gram matrix of a basis u_1, u_2 of P and $w = \big(u_1(x_0), u_2(x_0)\big)^\top$.

For the plane carrying the certificate this is settled by exact computation, with no appeal to any structure theory: the integer kernel of $(A - 20I)$ restricted to the 96 coordinates of W is computed in exact rational arithmetic, is two-dimensional, and yields $\|\operatorname{proj}_P(x_0)\|^2 = \frac{1}{48}$ on the nose, whence $\|v\|^2 = R^2 - \frac{1}{48} = \frac{5}{32} - \frac{1}{48} = \frac{13}{96}$.

The value is the same for every admissible x_0 and every such plane, and the reason is structural. Take u_1, u_2 of minimum weight 64 spanning P. Two minimum-weight words with distinct supports are ± 1-valued and meet in 32 positions, disagreeing on all of them, so $\Gamma = \begin{pmatrix} 64 & -32 \\ -32 & 64 \end{pmatrix}$ and $\Gamma^{-1} = \frac{1}{3072}\begin{pmatrix} 64 & 32 \\ 32 & 64 \end{pmatrix}$. For $x_0 \in W = \operatorname{supp}(u_1) \cup \operatorname{supp}(u_2)$ the vector w is one of $(\pm 1, 0)$, $(0, \pm 1)$, $(1, -1)$, $(-1, 1)$, and in each case $w^\top \Gamma^{-1} w = 64/3072 = \frac{1}{48}$. (The description of minimum-weight words used here is an empirically verified property of this code rather than a proved one; the counterexample itself does not depend on it, only the uniformity statement does. The sweep over all 4809 known planes confirms the value in every case.) $\square$

With $\|v\|^2 = \frac{13}{96}$, (2) reads $13\mu^2 + 2\mu - 17 = 0$, so

$$\mu = \frac{-1 + \sqrt{222}}{13} = 1.06920\ldots > 0.$$

Lemma 2. *For $j \in S$,*

$$\|y - x_j\|^2 = \begin{cases} \dfrac{1}{3} = b^2, & j \not\sim x_0, \\ \dfrac{1}{3} - \dfrac{\mu}{12} = \dfrac{53 - \sqrt{222}}{156} = 0.24423\ldots, & j \sim x_0. \end{cases}$$

In particular every new distance is at most b, with equality attained.

Proof. $x_j \in \operatorname{span} S$ and $x_0 - v \perp \operatorname{span} S$, so $\langle y, x_j \rangle = \mu \langle x_0, x_j \rangle$. Equation (2) gives $\|y\|^2 = \mu^2 \|v\|^2 = 2p_{\text{far}}\mu - 2p_{\text{far}} + R^2$, hence

$$\|y - x_j\|^2 = \|y\|^2 + R^2 - 2\mu\langle x_0, x_j\rangle = 2R^2 - 2p_{\text{far}} + 2\mu\big(p_{\text{far}} - \langle x_0, x_j\rangle\big).$$

If $j \not\sim x_0$ then $\langle x_0, x_j \rangle = p_{\text{far}}$ and the μ-term vanishes identically, leaving $2R^2 - 2p_{\text{far}} = b^2$. If $j \sim x_0$ then $\langle x_0, x_j \rangle = p_{\text{near}} = \frac{1}{32}$ and the μ-term equals $-2\mu(p_{\text{near}} - p_{\text{far}}) = -\mu/12 < 0$. $\square$

Theorem 3. *X is a set of 321 points in a 63-dimensional Euclidean space that cannot be partitioned into 64 subsets of smaller diameter. Hence Borsuk's conjecture is false in $\mathbb{R}^{63}$.*

Proof. $y \in \operatorname{span} S$, so X spans the 63-dimensional space $P^{\perp}$; its affine hull is 63-dimensional as well. By Lemma 2 the diameter of X is still b, attained (the vertex x_0 has non-neighbours in S).

Let $Y \subseteq X$ have diameter smaller than b. If $y \notin Y$ then Y is a set of points of S pairwise at distance a, i.e. a clique of $G[S]$, so $|Y| \le \omega(G[S]) = 5$. If $y \in Y$ then, by Lemma 2, $Y \setminus \{y\} \subseteq N(x_0) \cap S$ and is a clique of G; adjoining x_0 to it gives a clique of G, so $|Y \setminus \{y\}| \le \omega(G) - 1 = 4$ and again $|Y| \le 5$.

Therefore any partition of X into parts of smaller diameter has at least $\lceil 321/5 \rceil = 65$ parts, whereas Borsuk's conjecture in $\mathbb{R}^{63}$ asserts that 64 suffice. $\square$

Remark 4. X is a three-distance set, with squared distances $\frac{53-\sqrt{222}}{156}$, $\frac{1}{4}$ and $\frac{1}{3}$; the diameter graph has 39 120 edges. Leaving the two-distance category is exactly what makes the construction possible: within two-distance sets obtained from G the only move is deletion, and rank 63 forces $|S| \le 416 - 96 = 320 = 5 \cdot 64$.

Remark 5 (universality)*.* Lemma 1 involves no case analysis, so the construction works for each of the 96 choices of x_0 and for every 96-support plane. A sweep over all 4809 known planes gives a counterexample in every one of the 4809 cases, with no exceptions.

Several phantoms may be added simultaneously provided their sources form a clique $T \subseteq W$ of G: a smaller-diameter part is then $\{y_t : t \in I\} \cup K$ where $\{x_0^t : t \in I\} \cup K$ is a clique of G, so it still has at most 5 elements, and the diameter is unchanged. As $\omega(G) = 5$ this yields up to 325 points in $\mathbb{R}^{63}$ (verified). Sources that are *non*-adjacent are not admissible, for two reasons: some such pairs put their phantoms at squared distance $\mu^2/3 = (17 - 2\mu)/39 = 0.38106\ldots > \frac{1}{3}$ and break the diameter, and the rest raise the independence number to 6, because two phantoms from non-adjacent sources x, x' can share a part with a clique of $N(x) \cap N(x')$, and that common neighbourhood (the μ-graph of G, on 20 vertices) has clique number 4. Adding phantoms therefore costs about one unit of α per phantom beyond the first clique, and the counting bound is maximised at exactly the configuration used above.

Remark 6 (verification). Three independent programs check the example. (i) The construction itself. (ii) An adversarial audit that re-derives every step with disjoint code: the strong regularity and spectrum of G, $\omega(G) = 5$ by a fresh Bron–Kerbosch search, $\dim \operatorname{span} S = 63$ by exact rank computations modulo three primes, the exact two-dimensional kernel of $(A - 20I)|_W$ in rational arithmetic (giving $\|\operatorname{proj}_P(x_0)\|^2 = \frac{1}{48}$ with no appeal to the structure theory used in Lemma 1), the distances of Lemma 2 symbolically in $\mathbb{Q}(\sqrt{222})$, and $\alpha = 5$ for the diameter graph of X by Bron–Kerbosch on all 321 points. (iii) A stand-alone verifier that reads only the 321×63 matrix of coordinates and recomputes the affine dimension, the distance spectrum, the independence number of the diameter graph and the counting bound.

Remark 7 (the slack identity). The whole ladder is governed by one quantity. Delete the support Z of a j-dimensional subspace of $\mathcal{C}$, with $|Z| = d_j$ minimal, and add c phantoms sourced from a clique of Z. Then $|X| = v - d_j + c$, $\dim X = f - j$ and $\alpha = \omega$, so the counting bound refutes Borsuk in $\mathbb{R}^{f-j}$ exactly when

$$\operatorname{slack}(j) \;:=\; v - d_j + c - \omega\,(f - j + 1) \;>\; 0.$$

For $G_2(4)$ we have $v = 416$, $\omega = 5$, $f = 65$, and the measured generalized Hamming weights $d_1 = 64$, $d_2 = 96$, $d_3 = 128$, giving

$$\operatorname{slack}(0) = 86, \quad \operatorname{slack}(1) = 27 + c, \quad \operatorname{slack}(2) = c, \quad \operatorname{slack}(3) = c - 27.$$

The first is Bondarenko's $\mathbb{R}^{65}$, the second Jenrich's $\mathbb{R}^{64}$. The third is $\mathbb{R}^{63}$ and it is positive *if and only if* $c \geq 1$: the tie described above is the statement $\operatorname{slack}(2) = 0$ when no point may be added, and the phantom is worth exactly the one unit required. The fourth needs $c \geq 28$ while $c \leq \omega = 5$. The supports grow by 32 per step and the slack recovers only $\omega = 5$ per step, so the ledger goes bankrupt at $j = 3$; that, in one line, is why the record stops at 63.

Remark 8 (why 62 does not follow). The same criterion in $\mathbb{R}^{62}$ with c phantoms needs a three-dimensional subspace of $\mathcal{C}$ whose support has size at most $100 + c \leq 105$. Three facts rule that out.

First, a two-dimensional subspace spanned by two minimum-weight words has support $128 - m$ where m is the overlap, and over 1.2 million pairs the overlap takes only the values $8, 12, 16, 32$; so such supports have size 96, 112, 116 or 120, and *none lies in the window* $[97, 105]$. Second, a three-dimensional subspace of support at most 105 contains a two-dimensional one of support at most 105, which by the first fact must be exactly 96. So every candidate is $P + \langle c\rangle$ for a 96-support plane P, and its support is $96 + \operatorname{wt}(c|_{V\setminus W})$. Third, the restriction map $\mathcal{C} \to \mathbb{R}^{V\setminus W}$ has kernel exactly P, so its image is a 63-dimensional code of length 320, and the question becomes whether that code has a word of weight at most 9. Information-set decoding aimed at precisely this code returns 32 on every plane tried and never anything smaller, giving $d_3 = 128$ against the 105 required.

The margin is a factor of three and the case analysis is complete, but the first fact is an empirical property of this code rather than a proved one, and a failed search is evidence of a minimum weight, not a proof of it. So this is a strong obstruction rather than a theorem. The alternative of forcing $\omega(G[S]) = 4$ by deleting a hitting set of the 5-cliques is also excluded: after removing a 128-support there remain 72 192 five-cliques in the surviving 288 points, and the fractional hitting-set relaxation already needs 57.6 further deletions where only 35 are affordable.

Nor does mass phantom addition help. The 288-point rank-62 set does admit 128 phantom sources, and 6272 of the 8128 source pairs are diameter-compatible, so point count is not the obstacle; but phantoms from non-adjacent sources that are mutually near all fall into a single part, so α grows roughly one unit per phantom. Measured: $\alpha = 5$ at $|T| = 1$ (58 parts), 6 at $|T| = 2$, 7 at $|T| = 5$, 16 at $|T| = 16$. The best score over all $|T|$ is $58 - 63 = -5$: five parts short.

References

[1] A. V. Bondarenko, *On Borsuk's conjecture for two-distance sets*, Discrete Comput. Geom. 51 (2014), 509–515.

[2] T. Jenrich, A. E. Brouwer, *A 64-dimensional counterexample to Borsuk's conjecture*, Electron. J. Combin. 21(4) (2014), #P4.29.